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Scattering of Radiation by a Quasiperiodic Two-Dimensional Medium

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We study the scattering of radiation by a medium presenting inhomogeneities distributed in a quasiperiodic way. We show the existence of quasiperiodic solutions of the two-dimensional stationary wave equation, under certain conditions on the index of refraction, using a technique based on Dinaburg–Sinai method for one-dimensional Schrödinger equation with a quasiperiodic potential. Moreover we show that the energy spctrum contains a nonempty absolutely continuous component, with a subset having high degeneracy, provided the inhomogeneities are small enough.

KEY WORDS: Two-dimensional wave equation; quasiperiodic index of refraction; Dinaburg–Sinai method; Schrödinger equation; quasiperiodic solutions; absolutely continuous spectrum; degeneracy.

1. INTRODUCTION

There are many problems concerning the optical properties of inhomogeneous materials which arise in several fields, examples given in the study of the atmosphere, optical mineralogy, chemistry, etc. There can be many different causes for which a material is not optically homogeneous; the most common are anisotropies, crystal lattice dislocations, particulate inclusions, and so on (see Ref. 1 for an introduction to these problems). One way of modeling the propagation of light in inhomogeneous materials is to consider an index of refraction depending on spatial coordinates (see, e.g., Ref. 2); we will describe more precisely the scattering of radiation by particles distributed in a quasiperiodic way along

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one spatial direction. In such a case, the equation for stationary waves is the Helmholtz' one:[3]

$$\Delta\psi + \frac{1}{n^2} E\psi = 0 \tag{1.1}$$

where ψ is any component of magnetic or electric field, E is the square of the frequency of the wave, and $n = c(\mu_m \varepsilon_d)^{1/2}$ (index of refraction) is a quasiperiodic function of one spatial variable; here c is the speed of the light in the vacuum, μ_m is the absolute magnetic permeability, and ε_d is the absolute dielectric constant. The equation (1.1) is equivalent to the eigenvalue equation for a Laplace–Beltrami operator on a Riemanian manifold endowed with a quasiperiodic metric (Section 2). For the sake of simplicity, we have studied the case of a two-dimensional cylinder, but we believe that there is no difficulty in extending the result in the three-dimensional case, which corresponds to an optic fiber.

Let the refraction index n have spatial frequencies given by $\omega_1,..., \omega_\nu$ and satisfying a diophantine condition.[4] We show that there are many spatially quasiperiodic solutions with spatial frequencies λ, $\omega_1,..., \omega_\nu$ provided λ belongs to a certain Cantor set of positive Lebesgue measure. The values of E corresponding to these solutions are also given by a Cantor set $\mathscr{E}$ of positive Lebesgue measure. In other words, the medium is transparent only for the waves with frequencies in such a set.

Actually, this result is valid only if the modulation of n is not too large (nearly flat metric); and in this case the set $\mathscr{E}$ is concentrated in the low-frequency (infrared) region. Thus infrared waves are likely to cross the optic fiber, whereas ultraviolet waves are absorbed.

Mathematically, we first decompose the space of the solutions of the equation (1.1) into a direct sum of subspaces corresponding each to an eigenspace for the rotational symmetry around the axis of the fiber. The problem is then reduced to a family of one-dimensional equations indexed by an integer m representing the quantum number for the angular momentum (the infinitesimal generator of the rotational symmetry). The existence theorem is valid only for small values of m. Thus the modes with many nodes in the angular direction are unlikely to travel across the optic fiber. These one-dimensional equations have quasiperiodic coefficients. It is therefore possible to use the method of Kolmogoroff, Arnold, and Moser,[5] developed in classical mechanics for proving the stability of nearly integrable Hamiltonian systems. The method has been adapted to study the spectral properties of a Schrödinger equation in a quasiperiodic medium by Dinaburg and Sinai,[6] Belokolos,[7] and the estimates have been improved by Rüssmann.[8] We will actually use this latter work as a guideline. The Schrödinger equation with a quasiperiodic potential arose in physics for the following problems: a Bloch electron in a magnetic field, neglecting interband contributions,[9] the metal–insulator transition in organic conducting chains,[10] the stability of the motion of a particle in the gravitational field of several planets,[11] etc. For a review of results on this equation and the more general case of almost periodic potential, see, e.g., Ref. 12. In our problem, however, we need an additional analysis to estimate the range $\mathscr{E}$ of values of E which corresponds to the quasiperiodic solutions of (1.1). The main contribution of this paper is concentrated on this problem.

We also show that on $\mathscr{E}$ the Laplace–Beltrami operator has an absolutely continuous spectral measure. We remark that, to our knowledge, there are no examples of absolutely continuous spectra for two-dimensional problems connected to quasiperiodic Schrödinger equations. There are some results in the case of a band for the discrete Schrödinger equation with a stochastic potential (see, e.g., Ref. 13).

Moreover we show that, provided the refraction index is close enough to a constant, certain subsets of $\mathscr{E}$ with a positive Lebesgue measure correspond to a multiple spectrum. The corresponding degeneracy comes from the transmission of several rotational modes in the fiber.

This work is organized as follows: Section 2 is devoted to the introduction of the technical machinery and the rigorous statement of the results. In Section 3 we outline the proofs of these statements.

2. DEFINITIONS, HYPOTHESIS, AND RESULTS

In this section we set up notations and we state our results in precise form.

For $\nu \in \mathbb{N}$, and a, b in $\mathbb{R}^\nu$ or $\mathbb{C}^\nu$, we set

$$(a, b) = \sum_{i=1}^{\nu} a_i b_i, \qquad \|a\| = \sum_{i=1}^{\nu} |a_i|, \qquad |a| = \max_{i=1,\dots,\nu} |a_i|$$

We say that $\omega \in \mathbb{R}^\nu$ satisfies a diophantine condition[4] if there exists a Rüssmann approximation function (R.A.F.) Ω (see Ref. 8) such that

$$|(k\omega)| \geqslant \Omega(|k|) \qquad \forall k \in \mathbb{Z}^\nu \backslash \{0\} \tag{2.1}$$

As in Rüssmann,[8] we introduce, for $\delta > 0$,

$$\Phi_2(\delta) = \int_0^\infty \Omega\left(\frac{s}{\rho}\right)^{-2} e^{-s}\, ds$$

and

$$\Psi_2(\delta) = \inf \left\{ \prod_{l=0}^{\infty} \Phi_2(\delta_l)^{2^{-l-1}}, \delta_0 \geqslant \delta_1 \geqslant \cdots \geqslant \delta_l \geqslant \cdots 0; \sum_{l=0}^{\infty} \delta_l \leqslant \delta \right\}$$

We have[8,14]

$$\phi_2(\delta) < \Psi_2(\delta) < \infty \qquad \forall \delta > 0$$

Let ρ, R, and d be three positive constants. We define the sets

$$B(\rho) = \{z \in \mathbb{C}^{\nu}: |\text{Im } z| < \rho\}$$

$$\Lambda(\omega, \Omega, R) = \{\lambda \in \mathbb{C}: |\lambda - \tfrac{1}{2}(k\omega)| \geqslant \Omega(|k|), \forall k \in \mathbb{Z}^{\nu}; |\text{Im } \lambda| < R\}$$

$$D(d) = \{\mu \in \mathbb{C}: |\mu| < d\}$$

Now we define our manifold and we state the technical hypotheses on its quasiperiodic metric.

Let Q be the manifold:

$$Q = \{x, y: x \in \mathbb{R}, y \in \mathbb{T} \quad \mathbb{R}/\mathbb{Z}\}$$

endowed with the metric

$$g_{ij}(x, y) = \frac{\delta_{ij}}{F(x)}$$

where F: $\mathbb{R} \to \mathbb{R}$ is a continuous quasiperiodic function, bounded away from zero.

Let $\omega \in \mathbb{R}^{\nu}$ satisfy condition (2.1). We shall assume that there is v: $\mathbb{R} \to \mathbb{R}$ and $0 < \varepsilon \ll 1$ such that

$$\frac{1}{F(x)} = 1 - \varepsilon v(x) \tag{2.2}$$

Moreover, v will be a quasiperiodic function with mean zero and frequency module $\mathbb{Z}^{\nu}\omega$ ($\omega \in \mathbb{R}^{\nu}$). In addition, v will be analytic in the following sense: there exists $r > 0$ such that there is a holomorphic and 2π-periodic function

$$V: B(r) \subseteq \mathbb{C}^{\nu} \to \mathbb{C}$$

which is real for real arguments and satisfies

$$v(x) = V(\omega x) \qquad \forall x \in \mathbb{R}$$

$$\|v\|_r \equiv \sup_{z \in B(r)} |V(z)| < \infty$$

From the analyticity of v it follows that its Fourier coefficients are exponentially decreasing, namely,

$$v(x) = \sum_{k \in \mathbb{Z}^\nu \setminus \{0\}} v_k \exp\{i(k\omega)x\}$$

$$|v_k| \leqslant \exp(-r\|k\|) \qquad \forall k \in \mathbb{Z}^\nu \setminus \{0\}$$

Now we can define the Laplace–Beltrami operator Δ_Q induced by the metric $g_{ij}(x, y)$ on the Hilbert space $\mathscr{H} = L^2(Q, dxdy/F(x))$ by

$$(\Delta_Q \psi)(x, y) = F(x) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x, y)$$

$$\forall \psi \in \mathscr{D}(\Delta_Q) \subseteq \mathscr{H}$$

where $\mathscr{D}(\Delta_Q) = \mathscr{H}_2(Q, dx\, dy)$ is a Sobolev space of index 2.[15]

Then we set $H = -\Delta_Q$ and we find that the eigenvalue equation

$$H\psi = E\psi \tag{2.3}$$

is the same as the Helmholtz equation (1.1), if $n^2 = F(x)$. Before studying the solutions of this equation, we give the two following technical definitions:

Definition 2.1 (Lipschitz holomorphy). Let Λ be a closed set in $\mathbb{C}$ and $\mathscr{D}$ be an open domain in $\mathbb{C}^l$. Let $\mathscr{H}_b(\mathscr{D})$ be the space of holomorphic bounded functions on $\mathscr{D}$, endowed with the uniform topology.

Let f be a complex-valued function on $\Lambda \times \mathscr{D}$; f is called Lipschitz holomorphic if

$$\lambda \in \Lambda \to f(\lambda, \cdot) \in \mathscr{H}_b(\mathscr{D})$$

is a Lipschitz continuous from Λ into $\mathscr{H}_b(\mathscr{D})$ and it is holomorphic in the interior of Λ. We call $\mathscr{B}(\Lambda, \mathscr{D})$ the space of such functions.

Moreover, we call $\mathscr{B}(\Lambda)$ the space of complex-valued functions Lipschitz continuous function from Λ into $\mathscr{C}$, holomorphic in the interior of Λ. If endowed with the following norm, called Lipschitz norm,

$$f \in \mathscr{B}(\Lambda), \qquad \|f\|_L = \sup_{\lambda \in \Lambda} |f(\lambda)| + \sup_{\substack{\lambda, \lambda' \in \Lambda \\ \lambda \neq \lambda'}} \frac{|f(\lambda) - f(\lambda')|}{|\lambda - \lambda'|}$$

$\mathscr{B}(\Lambda)$ is a Banach space.

Definition 2.2 (interval and rectangle). If a, b are positive numbers and "q" an integer such that $|q| \leqslant a/4\pi b$, $J_q(a, b)$ is the open interval $]J_q^-(a, b), J_q^+(a, b)[$ where

$$J_q^{\pm}(a, b) = \frac{a}{2b} \pm \left(\frac{a^2}{4b^2} - 4\pi^2 b^2\right)^{1/2}$$

and $T_q(a, b)$ is the closed rectangle in the complex plane:

$$T_q(a, b) = \left\{\lambda \in \mathbb{C}: \operatorname{Re} \lambda \in \left[J_q^-(a, b), \frac{a}{2b}\right]; |\operatorname{Im} \lambda| \leqslant R_q(a, b)\right\}$$

where

$$R_q(a, b) = \left[J_q^+(a, b)^2 - \left(\frac{a}{2b}\right)^2\right]^{1/2}$$

Theorem 2.3 (existence of quasiperiodic solutions). Let Ω be a R.A.F., ω be a vector in $\mathbb{R}^\nu$ satisfying the diophantine condition (2.1). Let $F(x) = [1 - \varepsilon v(x)]^{-1}$ be chosen as in Eq. (2.2), with $\|v\|_r < +\infty$ for some $r > 0$.

Then we set

$$d(v) = \{2^{(3/2)\nu + 13}[1 + \Omega(0)]^{3/2}\, \Psi_2(r - r_\infty)\, \|v\|_r\}^{-1} \tag{2.4}$$

Giving $R > 0$, $0 < r_\infty < r$, $m_0 \in \mathbb{N}$, and $d > 0$ such that $m_0 \leqslant d(v)/4\pi d$, one can find (1) a sequence $0 < d_{m_0} \leqslant d_{m_0 - 1} \leqslant \cdots \leqslant d_0 \leqslant d$, (2) for each integer m, $0 \leqslant m \leqslant m_0$, a set $\Lambda_m = \Lambda(\omega, \Omega, R) \cap T_m(d(v), d)$ and a complex-valued function $E_m \in \mathscr{B}_m(\Lambda_m, D(d_m))$ such that, if $|\varepsilon| < d_m$, $\lambda \in \Lambda_m$, the equation (2.3) admits four linearly independent solutions $\psi^{\pm}_{\pm m}$ with

(a) $E = E_m(\lambda, \varepsilon)$

(b) $\psi^{+}_{\pm m}(\lambda, \varepsilon; x, y) = \chi(\lambda, \varepsilon; \omega x) \exp[i(\lambda x \pm 2\pi m y)]$

$\psi^{-}_{\pm m}(\lambda, \varepsilon; x, y) = \bar{\chi}(\bar{\lambda}, \bar{\varepsilon}; \omega x) \exp[-i(\bar{\lambda} x \mp 2\pi m y)]$

where χ is a complex-valued function in $\mathscr{B}(\Lambda_m, \{D(d_m) \times B(r_\infty)\})$ and is 2π periodic with respect to the variables in $B(r_\infty)$.

Remarks. (1) The number m_0 represents the maximal number of rotational modes which are almost periodic, according to our estimate.

(2) The smaller d the bigger m_0; the conclusion holds only if $|\varepsilon| < d$. Therefore the previous result is valid only for small disorder.

(3) The set Λ_m is concentrated in the low-energy region, and is a Cantor set. Points outside Λ_m but in the small rectangle $T_m(d(v), d)$ are

resonant points, for which a weird behavior is expected. On the other hand, λ represents a wave vector, and the equation $E = E_m(\lambda, \varepsilon)$ is the dispersion law in this medium. We think that E_m is also a weird function of λ having infinitely many absorption bands near the resonant points.

(4) This result is only a sufficient condition for the existence of such solutions. In particular, the numerical estimates are certainly not the optimal ones. However, as usual in this kind of problem, we believe that this theorem gives a qualitatively correct picture of what is going on. In particular, we hope that, apart from possible numerical improvements in the size of the several quantities involved, the medium is indeed transparent only in the infrared region, for only a finite number of rotational modes, and that it has a very intricate absorption spectrum.

(5) If we decrease $\Omega(0)$, we increase the set of ω's for which the result is valid; simultaneously we decrease $d(v)$, which means that we decrease the maximal value of the product εm_0 for which the previous theorem applies.

(6) The constant r_∞ represents a loss of analyticity for $\psi^{\pm}_{\pm m}$ compared to v. The closer r_∞ to r the bigger $\Psi_2(r - r_\infty)$ and the smaller $d(v)$.

(7) We remark also that $d(v)$, which in a sense measures the size of the highest value of ε for which our result holds, varies like $2^{-(3/2)\nu}$ as a function of ν. In the usual K.A.M. theorem, ν represents the number of degrees of freedom, and the critical value of the coupling varies like $(\nu!)^{-\sigma}$ for some $\sigma > 0$. This can be interpreted either as a special feature of the previous linear problem, or as the fact that the usual estimates are not the optimal ones.[16]

We see now that some additional hypotheses on the upper bound of ε, namely, d, and on the R.A.F. Ω imply a positive Lebesgue measure of the known part of the spectrum.

Proposition 2.4 (positive Lebesgue measure of the spectrum). If, besides all hypotheses of Theorem 2.3, the R.A.F. Ω is such that

$$\sum_{k \in \mathbb{Z}^\nu} \Omega(|k|) \leqslant \frac{1-\delta}{2}, \qquad \delta > 0 \tag{2.5}$$

and, given a nonnegative integer m, the constant d is such that

$$d \leqslant d(v)/2(1 + 4\pi^2 m^2)^{1/2} \tag{2.6}$$

then there is $0 < \tilde{d}_m \leqslant d_m$ such that, if $|\varepsilon| < \tilde{d}_m$, the image through $E_m(\cdot, \varepsilon)$ of $\Lambda_m \cap \mathbb{R}$ is a subset of positive Lebesgue measure of the spectrum of H.

Next we state the absolute continuity of the spectral measure and the existence of a set of degenerate energies of positive Lebesgue measure.

Theorem 2.5 (absolutely continuous spectrum). Let m be a non-negative integer. Under the hypotheses of Theorem 2.3 and Proposition 2.4, there exists $0 < \tilde{d}'_m < \tilde{d}_m$ such that, if $0 < |\varepsilon| < \tilde{d}'_m$, then the restriction of the spectral measure of H to the image through $E_m(\cdot, \varepsilon)$ of the set $\Lambda_m \cap \mathbb{R}$ is absolutely continuous.

Proposition 2.6 (degeneracy). Let N be a positive integer.
If, besides all hypotheses of Theorem 2.3, the R.A.F. Ω is such that

$$\sum_{k \in \mathbb{Z}^\nu} \Omega(|k|) < \frac{1}{(18\pi)^2 N^3} \tag{2.7}$$

and if

$$d = d(v)/2(1 + 4\pi^2 N^2)^{1/2} \tag{2.8}$$

then there exists a subset of positive Lebesgue measure of the known part of the spectrum of H with degeneracy $2(2N+1)$ provided that

$$|\varepsilon| < \tilde{d}^{(N)} \equiv \min_{m = 0,\dots,N} \{\tilde{d}_m\}$$

3. SKETCH OF THE PROOFS

3.1. Proof of Theorem 2.3

We divide this proof into three steps:

Step 1. We show that, for studying equation (2.3), we can restrict ourselves to the study of ordinary differential equations (3.1) below, parametrized by $m \in \mathbb{Z}$, owing to an invariant decomposition of the Hilbert space $\mathscr{H}$ under the action of the Laplace-Beltrami operator.

Step 2. As usual [6, 8] we replace the Schrödinger-like equation (3.1) below by a first order system. The existence of quasi-periodic solutions of this system is insured by the Dinaburg-Sinai theorem [6, 7, 8].

Step 3. The set of energies E for which such solutions exist is given by the range of a function, solution of an implicit equation (see (3.7) and (3.8) below). We analyze this equation to show that it admits solutions for ε "small."

Step 1 (passage to an ordinary differential equation). From a direct computation we can see that the space $\mathscr{H} = L^2(B, dx\, dy/F(x))$ is the direct sum of the eigenspaces

$$\mathscr{H}_m = \left\{\psi \in \mathscr{H} : \psi(x, y) = \varphi(x)\, e^{2\pi i m y};\ m \in \mathbb{Z},\ \varphi \in L^2\left(\mathbb{R}, \frac{dx}{F(x)}\right)\right\}$$

of the one-parameter group of y rotations:

$$\{U(\tau)\}_{\tau\in\mathbb{T}}: U(\tau)\,\psi(x, y)=\psi(x, y-\tau) \qquad \forall\psi\in\mathscr{H}$$

and the operator Δ_Q leaves the eigenspaces $\mathscr{H}_m$ of $\{U(\tau)\}_{\tau\in\mathbb{T}}$ invariant, that is,

$$\Delta_Q \restriction (\mathscr{D}\cap\mathscr{H}_m)\subseteq\mathscr{H}_m \qquad \forall m\in\mathbb{Z}$$

Now we set $H_m=-\Delta_Q \restriction (\mathscr{D}\cap\mathscr{H}_m)$; then we may restrict ourselves to the study of H_m's spectra, i.e., of the following ordinary differential equation, parametrized by the integer m:

$$-\frac{d^2}{dx^2}\,\varphi(x)+\varepsilon E v(x)\,\varphi(x)=(E-4\pi^2m^2)\,\varphi(x) \tag{3.1}$$

We remark that m and $-m$ give the same equation; then we are allowed to consider m as a nonnegative integer in the following.

Step 2 (quasiperiodic solutions). Let us consider the system

$$\begin{aligned}\frac{dX(x)}{dx}&=[\zeta J+\mu Q(z)]\,X(x)\\ \frac{dz}{dx}&=\omega\end{aligned} \tag{3.2}$$

where

$$z\in B(r)\subseteq\mathbb{C}^\nu; \qquad Q(z)=\begin{pmatrix}0 & 0\\ V(z) & 0\end{pmatrix}; \qquad J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}; \qquad \zeta,\mu\in\mathbb{C}$$

$$X(x)=\begin{pmatrix}X^{(1)}(x)\\ X^{(2)}(x)\end{pmatrix}$$

$\omega\in\mathbb{R}^\nu$ satisfies the diophantine condition (2.1) and V is the bounded holomorphic extension of v to $B(r)$ (see Section 2).

This system is equivalent to Eq. (3.1), provided

$$\begin{aligned}&X^{(1)}(x)=\varphi(x); \qquad X^{(2)}(x)=\frac{d\varphi/dx}{(E-4\pi^2m^2)^{1/2}}\\ &\zeta=(E-4\pi^2m^2)^{1/2}; \qquad \mu=\varepsilon\frac{E}{(E-4\pi^2m^2)^{1/2}}\end{aligned} \tag{3.3}$$

The following theorem[6-8] gives an existence result for (3.2).

Theorem 3.1 (Dinaburg-Sinai, Belokolos, Rüssmann). Let r_∞ be $0 < r_\infty < r$, let $d(v)$ be given by (2.4). There exists a function

$$a\colon (\lambda; \mu) \in A(\omega, \Omega, R) \times D(d(v)) \to \mathbb{C}$$

with the following properties: (i) $a \in \mathscr{B}(A(\omega, \Omega, R), D(d(v)))$ and (ii) $a(\lambda, \mu) \in \mathbb{R}$ if λ and μ are real; $a(\lambda, 0) = 0$; such that, if $\lambda \in A(\omega, \Omega, R)$, $\mu \in D(d(v))$ and if ζ satisfies

$$\zeta = \lambda + a(\lambda, \mu) \tag{3.4}$$

then there exist two linearly independent solutions of the system (3.2) of the form

$$\begin{aligned} X_+(\lambda, \mu; x) &= e^{i\lambda x}\chi(\lambda, \mu; \omega x) \\ X_-(\lambda, \mu; x) &= e^{-i\lambda x}\bar{\chi}(\bar{\lambda}, \bar{\mu}; \omega x) \end{aligned} \tag{3.5}$$

where

$$\chi(\lambda, \mu; \quad) = \begin{bmatrix} \chi(\lambda, \mu; z) \\ \chi'(\lambda, \mu; z) \end{bmatrix} \tag{3.6}$$

with χ, χ' belonging to $\mathscr{B}(A(\omega, \Omega, R), \{D(d(v)) \times B(r_\infty)\})$ and 2π periodic in $B(r_\infty)$.

A detailed version of the proof of this theorem is contained in Ref. 17, where we have followed Rüssman's point of view,[8,18,19] using the notations of Ref. 14. Given the system (3.2), the idea is to define a succession of changes of the variable X, 2π-periodic in x and μ-depending, such that, after infinitely many steps, this system is transformed into another one, whose solutions are known. This idea goes back to Bogoliubov and Krylov,[20] it was precisely stated by Kolmogorov,[5b] and the convergence follows the lines developed by Arnold[5a] and Moser.[5c]

Step 3. We see, from (3.3), (3.4) and the definition of the function "a," that the set of values of E for which the existence of quasiperiodic solutions holds through Theorem 3.1 is the range of the solutions of the following implicit equation:

$$[E_m(\lambda, \varepsilon) - 4\pi^2 m^2]^{1/2} = \lambda + a\left(\lambda, \varepsilon \frac{E_m(\lambda, \varepsilon)}{[E_m(\lambda, \varepsilon) - 4\pi^2 m^2]^{1/2}}\right) \tag{3.7}$$

with the condition

$$\left| \varepsilon \frac{E_m(\lambda, \varepsilon)}{[E_m(\lambda, \varepsilon) - 4\pi^2 m^2]^{1/2}} \right| \leqslant d(v) \tag{3.8}$$

We shall now state the existence of solutions for this equation, provided ε and m are small enough. More precisely, we get the following:

Proposition 3.2. Let m_0 be an integer such that $0 \leqslant m_0 \leqslant d(v)/4\pi d$. For each $0 \leqslant m \leqslant m_0$, $m \in \mathbb{Z}$, there is a constant $0 < d_m \leqslant d$ such that there exists a complex-valued function E_m belonging to $\mathscr{B}(\Lambda_m, D(d_m))$ [where $\Lambda_m = \Lambda(\omega, \Omega, R) \cap T_m(d(v), d)$; see Definition 2.2], real for real arguments, and satisfying the condition (3.8) and the equation (3.7).

This result will follow from the implicit function theorem in the holomorphic case.

Indeed, let d' be such that $d(v) < d' < (7/3)^{1/2} d(v)$. $L_m(d', d)$ will be the open not empty subset of $\mathscr{B}(\Lambda_m)$ made up by the functions with absolute value in $J_m(d', d)$. For $\varepsilon \in D(d)$ and $f \in L_m(d', d)$, the function

$$F_m(\varepsilon, f) = \left[f - a\left(\cdot, \varepsilon \frac{f^2 + 4\pi^2 m^2}{f}\right) - id_{\Lambda_m} \right] \in \mathscr{B}(\Lambda_m)$$

is well defined.

Thus it is possible to conclude that there are $0 < d_m < d$ and $u_m \in \mathscr{B}(\Lambda_m, D(d_m))$ with the properties

$$u_m(\cdot, \varepsilon) \in \mathbf{L}_m(d', d); \qquad F_m(\varepsilon, u_m(\cdot, \varepsilon)) = 0 \qquad \forall \varepsilon \in D(d_m)$$

and

$$u_m(\cdot, 0) = id_{\Lambda_m}$$

Therefore the solution of the equations (3.7), (3.8) is given by

$$E_m(\lambda, \varepsilon) = u_m(\lambda, \varepsilon)^2 + 4\pi^2 m^2 \tag{3.9}$$

Proof of Proposition 2.4. In order to simplify the notations, in the next we set

$$\Lambda_{\mathbb{R}} = \Lambda(\omega, \Omega, R) \cap \mathbb{R}, \qquad I_m = T_m(d(v), d) \cap \mathbb{R}$$

It is sufficient to show that the image of $\Lambda_{\mathbb{R}} \cap I_m$ by $u_m(\cdot, \varepsilon)$ has a positive Lebesgue measure [see (3.9)]. In the following we sometimes omit the dependence on ε.

(1) We extend $u_m \upharpoonright (\Lambda_{\mathbb{R}} \cap I_m)$ to a function $\hat{u}_m$ defined on the whole interval I_m in the following way: $\Lambda_{\mathbb{R}}$ is a Cantor set and then there exists a countable family $\{L_q\}_{q \in \mathbb{N}}$ of open disjoint intervals such that

$$\rho(\Lambda_{\mathbb{R}}) \cap I_m = \bigcup_{q \in \mathbb{N}} L_q$$

where $\rho(B) = \mathbb{R} \backslash B$ with $B \subseteq \mathbb{R}$.

For $\lambda \in L_q =]a_q, b_q[$ with a_q, $b_q \in \Lambda_{\mathbb{R}} \cap I_m$, we define $\hat{u}_m(\lambda)$ by the linear interpolation:

$$\hat{u}_m(\lambda) = \frac{u_m(b_q)(\lambda - a_q) - u_m(a_q)(\lambda - b_q)}{b_q - a_q}$$

(2) Let $|A|$ be the Lebesgue measure of $A \subseteq \mathbb{R}$. We remark that

$$|\Lambda_{\mathbb{R}} \cap I_m| \geqslant |I_m| - |\rho(\Lambda_{\mathbb{R}})|$$

and

$$|\rho(\Lambda_{\mathbb{R}})| \leqslant \sum_{k \in \mathbb{Z}^\nu} 2\Omega(|k|)$$

But (2.5) and (2.6) imply, respectively, that

$$|\rho(\Lambda_{\mathbb{R}})| \leqslant 1 - \delta$$

and

$$|I_m| > 1 \tag{3.10a}$$

Then we have

$$|\Lambda_{\mathbb{R}} \cap I_m| \geqslant \delta > 0 \tag{3.10b}$$

(3) We define the following family of subsets of I_m:

$$\mathscr{A} = \{A \subseteq I_m : |\hat{u}_m(A)| \geqslant \tilde{K}|A|\}$$

where $\tilde{K}$ is a positive constant to be determined below. We show that every interval belongs to $\mathscr{A}$. Through the Lipschitz holomorphy of $a(\lambda, \mu)$ in $\Lambda(\omega, \Omega, R)\, D(d(v))$, the Schwarz principle, the mean value theorem and the Cauchy integral formula, it is possible to show that there exist two positive constants $\mathbf{d}_m \leqslant d_m$ and $K(m)$, depending on d, d', $d(v)$ and m, such that if $|\varepsilon| < \mathbf{d}_m$:

$$\|u_m(\cdot, \varepsilon) - id_{\Lambda_m}\|_L \leqslant K(m)\,|\varepsilon| \tag{3.11a}$$

Now if

$$|\varepsilon| < \tilde{d}_m \equiv \min\{\mathbf{d}_m, \tfrac{1}{2}K(m)\} \tag{3.11b}$$

we have

$$\|\hat{u}_m^{-1}(\cdot, \varepsilon)\|_L \leqslant 2\,\|id_{\Lambda_m}\|_L$$

and so, for each $a, b \in I_m$

$$|\hat{u}_m(b) - \hat{u}_m(a)| \geqslant \frac{|b-a|}{2\,\|id_{\Lambda_m}\|_L}$$

i.e., every interval belongs to $\mathscr{A}$, with $\tilde{K} = 1/2\,\|id_{\Lambda_m}\|_L$. It is easy to show that $\mathscr{A}$ contains also all denumerable unions and intersections of intervals; then it contains all Borel subsets of I_m; in particular, the closed set $\Lambda_{\mathbb{R}} \cap I_m$ will belong $\mathscr{A}$; then we have

$$|\hat{u}_m(\Lambda_{\mathbb{R}} \cap I_m)| > 0$$

from the definition of $\mathscr{A}$ and (3.10b).

Proof of Theorem 2.5. The guide line of this proof follows Refs. 6 and 14. We sometimes omit the dependence on ε.

Let h be a vector in $L^2(\mathbb{R}, dx) \cap L^1(\mathbb{R}, dx)$; let $\underline{\lambda} > 0$ and $\beta > 0$ be such that $\underline{\lambda} \pm i\beta \in \Lambda_m$; let $G_{E_m(\underline{\lambda} + i\beta)}$ be the resolvent of H_m at the point $E_m(\underline{\lambda} + i\beta)$. Using the Wronskian formula for the resolvent (6, 14) and some estimates on the fundamental solutions of the equation (3.1) (17), we can prove that there is a constant $K^{(m)}$ depending on d, $d(v)$ and m, such that

$$|\langle h, G_{E_m(\underline{\lambda} + i\beta)} h \rangle| \leqslant K^{(m)}\,\|h\|_{L^1(\mathbb{R}, dx/F(x))}\,\|h\|_{L^2(\mathbb{R}, dx)}$$

Then we set

$$\tilde{K}_m(h) \equiv K^{(m)}\,\|h\|_{L^1(\mathbb{R}, dx/F(x))}\,\|h\|_{L^2(\mathbb{R}, dx)}$$

Now we define two real functions ξ_m^ε and η_m^ε by

$$\xi_m^\varepsilon(\underline{\lambda}, \beta) \pm i\eta_m^\varepsilon(\underline{\lambda}, \beta) = E_m(\underline{\lambda} \pm i\beta, \varepsilon) - E_m(\underline{\lambda}, \varepsilon)$$

If σ_h is the spectral measure of H_m corresponding to the vector h previously defined, we have

$$\int \frac{\eta_m(\underline{\lambda}, \beta)\,\sigma_h(dE)}{[E - E_m(\underline{\lambda}) - \xi_m(\underline{\lambda}, \beta)]^2 + \eta_m(\underline{\lambda}, \beta)^2} \leqslant 2\tilde{K}_m(h) \tag{3.12}$$

because the left hand side is twice the imaginary part of

$$\langle h, G_{E_m(\underline{\lambda} + i\beta)} h \rangle$$

By a direct computation from the relation (3.11a), it is possible to show that there is a constant $d'_m \leqslant d_m$ and three constants $K_{1,2,3}$, depending only on $\underline{\lambda}$, such that, if $|\varepsilon| < d'_m$ and $\beta < 1$

$$|\xi_m(\underline{\lambda}, \beta)| \leqslant K_1 \beta$$
$$0 < K_2 \beta \leqslant \eta_m(\underline{\lambda}, \beta) \leqslant K_3 \beta$$

These estimates and the relation (3.12) allow us to conclude that if I_β is an interval of Lebesgue measure 2β, centered at $E_m(\underline{\lambda})$ there is a constant $\bar{K}$ given by

$$\bar{K} = 2K_2/[(1+K_1)^2 + K_3^2]$$

such that

$$\sigma_h(I_\beta) \leqslant \frac{\tilde{K}_m(h)}{\bar{K}} |I_\beta|$$

which insures the absolute continuity of the spectral measure σ_h.

Proof of Proposition 2.6. In what follows we use the notations stated in the proof of the proposition 2.4.

We show that the spectrum is degenerated in the following sense: There exist $\lambda_0, \lambda_1, ..., \lambda_N \in \Lambda_{\mathbb{R}}$ and $\tilde{d}^{(N)} > 0$ such that, if $|\varepsilon| < \tilde{d}^{(N)}$, then $\forall m = 0, ..., N$, $E_m(\lambda_m, \varepsilon)$ is an allowed value for the energy E in the corresponding equation (3.1) and moreover:

$$[E_m(\lambda_m) - 4\pi^2 m^2]^{1/2} = [E_{m'}(\lambda_{m'}) - 4\pi m'^2]^{1/2}$$
$$\forall m, m' = 0, ..., N$$

We must then show that the Lebesgue measure of the intersection of the images of the allowed sets by u_m, for $m = 0, ..., N$, is positive.

Thanks to (2.8), we get (3.10a), $\forall m = 0, ..., N$. Moreover,

$$I_m \subseteq I_l \qquad \text{if} \quad m > l$$

So we can fix an interval I such that

$$|I| = 1 \qquad \text{and} \qquad I \subseteq I_m \qquad \forall m = 0, ..., N$$

Now, the shape of the set Λ_m and (2.8) give

$$|\hat{u}_m(I)| \geqslant 1/18\pi N \qquad \forall m = 0, ..., N \tag{3.13a}$$

if

$$|\varepsilon| < \tilde{d}^{(N)} \equiv \min_{m=0,...,N} \{d_m\} \tag{3.13b}$$

The remark that $\hat{u}_m(\lambda) \in J_m(d', d)\ \forall \lambda \in I_m$ and $J_m(d', d) \subseteq J_l(d', d)$ if $m > l$, and (3.13a), allow us to fix another interval U such that

$$|U| = \frac{1}{18\pi N} \qquad \text{and} \qquad U \subseteq \hat{u}_m(I) \qquad \forall m = 0,..., N$$

In what follows we set

$$A \equiv \bigcap_{m=0}^{N} \hat{u}_m(\Lambda_{\mathbb{R}} \cap I), \qquad \rho(A) = \bigcup_{m=0}^{N} \rho(\hat{u}_m(\Lambda_{\mathbb{R}} \cap I)) \equiv \bigcup_{m=0}^{N} V_m$$

Our aim is to show that $|A| > 0$. This will be achieved if we prove

$$|V_m \cap U| < \frac{1}{N} |U| \tag{3.14}$$

Indeed, as

$$|A| \geqslant |U| - \sum_{m=0}^{N} |V_m \cap U|$$

(3.14) will imply

$$|A| > |U| - N\left(\frac{1}{N} |U|\right) = 0$$

Then, in order to show (3.14), we remark that

$$V_m \cap U \subseteq u_m(\rho(\Lambda_{\mathbb{R}}) \cap I) \qquad \forall m = 0,..., N \tag{3.15}$$

But (3.11a), (3.11b), and (3.13b) give

$$|u_m(\rho(\Lambda_{\mathbb{R}}) \cap I)| \leqslant \tfrac{1}{2} + \|id_{\Lambda_m}\|_L |\rho(\Lambda_{\mathbb{R}}) \cap I|$$

and from (2.8), (3.10b), (2.7), and (3.15) we may conclude

$$|V_m \cap U| < 9\pi N \cdot 2 \frac{1}{(18\pi)^2 N^3} = \frac{|U|}{N}$$

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